

PERTURBATIONS OF VON NEUMANN SUBALGEBRAS WITH FINITE INDEX

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ABSTRACT. In this paper, we study uniform perturbations of von Neumann subalgebras of a von Neumann algebra. Let N and M be von Neumann subalgebras of a von Neumann algebra with finite probabilistic index in the sense of Pimsner-Popa. If N and M are sufficiently close, then N and M are unitarily equivalent. The implementing unitary can be chosen as being close to the identity.

1. INTRODUCTION

In 1972, the uniform perturbation theory of operator algebras was started by Kadison and Kastler [14]. They defined a metric on the set of operator algebras on a fixed Hilbert space by the Hausdorff distance between their unit balls. We get basic examples of close operator algebras by small unitary perturbations. Namely, given an operator algebra $N \subset \mathbb{B}(H)$ and a unitary operator $u \in \mathbb{B}(H)$, if u is close to the identity operator, then uNu^* is close to N . Conversely, Kadison and Kastler suggested that suitably close operator algebras must be unitarily equivalent. This conjecture was solved positively for injective von Neumann algebras in [4, 23, 11] with earlier special cases [3, 17]. Recently, [2] gave classes of non-injective von Neumann algebras for which this conjecture is valid. In [5], for von Neumann subalgebras in a *finite* von Neumann algebra, Kadison-Kastler conjecture was solved positively. However, for general von Neumann algebras, this conjecture is still open.

Examples of non-separable C^* -algebras which are arbitrarily close but non-isomorphic were found in [1]. However, for general separable C^* -algebras, Kadison-Kastler conjecture is still open. In [8], the conjecture was solved positively for separable nuclear C^* -algebras. Earlier special cases of [8] was studied in [6, 18, 19, 15]. The author and Watatani showed that for an inclusion of simple C^* -algebras with finite index, sufficiently close intermediate C^* -subalgebras are unitarily equivalent in [10]. Although our constants depend on inclusions, Dickson got universal constants independent of inclusions in [9].

In this paper, we study uniform perturbations of von Neumann subalgebras of a von Neumann algebra with finite index. Let N and M be von Neumann subalgebras of a von Neumann algebra L with conditional expectations $E_N : L \rightarrow N$ and $E_M : L \rightarrow M$ of finite probabilistic index in the sense of Pimsner-Popa [20]. If the distance between N and M is sufficiently small, then N and M are unitarily equivalent. Moreover, the implementing unitary can be chosen as being close to the identity. In general, there exist examples of arbitrarily close unitarily conjugate C^* -algebras where the implementing unitaries could not be chosen to be close to the identity in [12]. Compared with the author and Watatani's C^* -algebraic case [10], we do not assume that N and M have a common subalgebra with finite index.

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2. DISTANCE AND THE RELATIVE DIXMIER PROPERTY

In this paper, all von Neumann algebras are countably decomposable, that is, they have faithful normal states.

We recall the distance defined by Kadison and Kastler in [14] and near inclusions defined by Christensen in [6]. For a von Neumann algebra N , we denote by N_1 the unit ball of N .

Definition 2.1. Let N and M be von Neumann algebras in $\mathbb{B}(H)$. Then, the distance between N and M is defined by

$$d(N, M) := \max \left\{ \sup_{n \in N_1} \inf_{m \in M_1} \|n - m\|, \sup_{m \in M_1} \inf_{n \in N_1} \|m - n\| \right\}.$$

Let $\gamma > 0$. We say that N is γ contained in M and write $N \subseteq_\gamma M$ if for any $n \in N_1$, there exists $m \in M$ such that $\|n - m\| \leq \gamma$.

If $d(N, M) < \gamma$, then for any x in either N_1 or M_1 , there exists y in the other unit ball such that $\|x - y\| \leq \gamma$.

The following well-known fact is needed to show that maps are onto in Proposition 3.1.

Lemma 2.2. Let N and M be von Neumann algebras in $\mathbb{B}(H)$. If $N \subset M$ and $d(N, M) < 1$, then $N = M$.

The next lemma records some standard estimates.

Lemma 2.3. Let A be a unital C^* -algebra.

- (1) Let $x \in A$ satisfy that $\|x - I\| < 1$ and $u \in A$ be the unitary in the polar decomposition $x = u|x|$. Then,

$$\|u - I\| \leq \sqrt{2}\|x - I\|.$$

- (2) Let p and q be projections in A with $\|p - q\| < 1$. Then, there exists a unitary $w \in A$ such that

$$wpw^* = q \quad \text{and} \quad \|w - I\| \leq \sqrt{2}\|p - q\|.$$

Jones introduced index for inclusions of type II_1 factors in [13]. For arbitrary factors, Kosaki extended Jones' notion of index in [16]. The following definition was introduced by Pimsner and Popa in [20].

Definition 2.4. Let $N \subset M$ be an inclusion of von Neumann algebras and $E : M \rightarrow N$ be a conditional expectation. Then, we call E is of *finite probabilistic index* if there exists $c \geq 0$ such that

$$E(x^*x) \geq cx^*x \quad \text{for } x \in M.$$

When E is of finite probabilistic index, we define the probabilistic index of E by $(\sup\{c \geq 0 \mid E(x^*x) \geq cx^*x \text{ for } x \in M\})^{-1}$.

We recall the basic construction (see [21]). Let $N \subset M$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation $E_N : M \rightarrow N$ and ψ be a faithful normal state on N . Put $\phi := \psi \circ E_N$. Then, ϕ is a faithful normal state on M . Let (H, π, ξ) be the GNS triplet associated with ϕ . Then, we get the Jones projection $e_N \in \mathbb{B}(H)$ satisfying

$$\text{Im}(e_N) = [N\xi] \quad \text{and} \quad e_N(x\xi) = E_N(x)\xi \quad \text{for } x \in M.$$

The basic construction $\langle M, e_N \rangle$ is the von Neumann algebra in $\mathbb{B}(H)$ generated by M and e_N . If E_N is of finite probabilistic index, then there exists a conditional expectation $E_M : \langle M, e_N \rangle \rightarrow M$ of finite probabilistic index by [21].

Let $N \subset M$ be an inclusion of von Neumann algebras. For any $x \in M$, we will denote by $C_N(x)$ the norm closure of the convex hull of $\{uxu^* \mid u \text{ is unitary element in } N\}$. We recall the relative Dixmier property for inclusions of von Neumann algebras after Popa [22].

Definition 2.5. Let $N \subset M$ be an inclusion of von Neumann algebras. Then, we say that $N \subset M$ has the *relative Dixmier property* if for any $x \in M$, $C_N(x) \cap N' \cap M \neq \emptyset$.

In [22], Popa showed the following theorem.

Theorem 2.6 (Popa [22]). *Let $N \subset M$ be an inclusion of von Neumann algebras with a conditional expectation $E : M \rightarrow N$ of finite probabilistic index. Then, $N \subset M$ has the relative Dixmier property.*

We shall show relations between the relative Dixmier property and the distance.

Let $N \subset M$ be an inclusion of von Neumann algebras. For any $x \in M$, the map $ad(x) : N \rightarrow M$ is defined by $(ad(x))(y) = yx - xy$.

The proof of the next proposition follows from [7, Proposition 2.5].

Proposition 2.7. *Let N and M be von Neumann subalgebras of a von Neumann algebra L with $N \subseteq_\gamma M$. If $N \subset L$ has the relative Dixmier property, then*

$$M' \cap L \subseteq_{2\gamma} N' \cap L.$$

Proof. For any $x \in M' \cap L_1$, there exists $y \in C_N(x) \cap N' \cap M$. Since for any unitary $u \in N$,

$$\|uxu^* - x\| = \|ux - xu\| = \|(ad(x))(u)\| \leq \|ad(x)\|,$$

we have $\|y - x\| \leq \|ad(x)\|$. On the other hand, for any $n \in N_1$, there exists $m \in M$ such that $\|n - m\| \leq \gamma$. Thus,

$$\begin{aligned} \|(ad(x))(n)\| &= \|nx - xn\| = \|nx - mx + xm - xn\| \\ &\leq \|n - m\| + \|m - n\| \leq 2\gamma. \end{aligned}$$

Namely, $\|x - y\| \leq \|ad(x)\| \leq 2\gamma$. □

3. PERTURBATIONS

In the following proposition, we construct a surjective $*$ -isomorphism between von Neumann subalgebras of a von Neumann algebra with finite probabilistic index. The argument is originated in early work of Christensen [4, 5].

Proposition 3.1. *Let N and M be von Neumann subalgebras of a von Neumann algebra L with conditional expectations $E_N : L \rightarrow N$, $E_M : L \rightarrow M$ of finite probabilistic index. If $d(N, M) < 1/15$, then there exists a normal surjective $*$ -isomorphism $\Phi : N \rightarrow M$ such that $\|\Phi - id_N\| < 14d(N, M)$.*

Proof. Put $\gamma := (1.01)d(N, M)$. Let $\langle L, e_M \rangle$ be the basic construction by using $E_M : L \rightarrow M$. Then, there exists a conditional expectation $E_L : \langle L, e_M \rangle \rightarrow L$ of finite probabilistic index. Since

$E_N \circ E_L : \langle L, e_M \rangle \rightarrow N$ is of finite probabilistic index, $N \subset \langle L, e_M \rangle$ has the relative Dixmier property by Theorem 2.6. Therefore,

$$M' \cap \langle L, e_M \rangle \subseteq_{2\gamma} N' \cap \langle L, e_M \rangle$$

by Proposition 2.7. Thus, there exists $t \in N' \cap \langle L, e_M \rangle$ such that $\|t - e_M\| \leq 2\gamma < 1/2$. Put $p := \chi_{[1-2\gamma, 1+2\gamma]}((t + t^*)/2)$. Since we have $\|p - e_M\| \leq \|p - t\| + \|t - e_M\| \leq 4\gamma < 1$, there exists a unitary $w \in \langle L, e_M \rangle$ such that

$$we_M w^* = p \quad \text{and} \quad \|w - I\| \leq 4\sqrt{2}\gamma$$

by Lemma 2.3. For any $x \in N$, we define $\tilde{\Phi}(x) := e_M w^* x w e_M = w^* p x p w$. Then, $\tilde{\Phi} : N \rightarrow e_M \langle L, e_M \rangle e_M$ is a normal $*$ -homomorphism, because $p \in N'$. Now, there exists a surjective $*$ -isomorphism $\iota : e_M \langle L, e_M \rangle e_M \rightarrow M$. Hence, we can define a normal $*$ -homomorphism $\Phi := \iota \circ \tilde{\Phi} : N \rightarrow M$. For any $x \in N_1$,

$$\begin{aligned} \|\Phi(x) - E_M(x)\| &= \|e_M(\Phi(x) - E_M(x))e_M\| \\ &= \|e_M w^* x w e_M - e_M x e_M\| \\ &\leq 2\|w - I\| \leq 8\sqrt{2}\gamma. \end{aligned}$$

Therefore, by [10, Lemma 3.2],

$$\|\Phi - id_N\| \leq \|\Phi - E_M|_N\| + \|E_M|_N - id_N\| \leq (8\sqrt{2} + 2)\gamma < 14d(N, M) < 1.$$

This gives that Φ is a $*$ -isomorphism.

Moreover, for any $x \in M_1$, there exists $y \in N_1$ such that $\|x - y\| \leq \gamma$. Then,

$$\begin{aligned} \|x - \Phi(y)\| &\leq \|x - y\| + \|y - \Phi(y)\| \\ &\leq \gamma + (8\sqrt{2} + 2)\gamma < 15d(N, M) < 1. \end{aligned}$$

Since this gives that $d(M, \Phi(N)) < 1$, $\Phi(N) = M$ by Lemma 2.2. \square

The following is our main theorem in this paper. We based on Christensen's work [4, Proposition 4.2] and [5, Proposition 3.2].

Theorem 3.2. *Let N and M be von Neumann subalgebras of a von Neumann algebra L with conditional expectations $E_N : L \rightarrow N$, $E_M : L \rightarrow M$ of finite probabilistic index. If $d(N, M) < 1/15$, then there exists a unitary $u \in L$ such that $N = uMu^*$ and $\|u - I\| < 20d(N, M)$.*

Proof. By Proposition 3.1, there exists a normal surjective $*$ -isomorphism $\Phi : N \rightarrow M$ such that $\|\Phi - id_N\| < 14d(N, M)$. Put

$$K := \left\{ \begin{pmatrix} x & 0 \\ 0 & \Phi(x) \end{pmatrix} \mid x \in N \right\}.$$

Then, we can define a conditional expectation $E_K : \mathbb{M}_2(L) \rightarrow K$ of finite probabilistic index by

$$E_K \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \frac{E_N(a) + \Phi^{-1}(E_M(d))}{2} & 0 \\ 0 & \frac{\Phi(E_N(a)) + E_M(d)}{2} \end{pmatrix} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(L).$$

Therefore, $K \subset \mathbb{M}_2(L)$ has the relative Dixmier property by Theorem 2.6. Applying the relative Dixmier property for $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(L)$, we obtain x in $C_K \left(\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \right) \cap K' \cap \mathbb{M}_2(L)$. Then, there

exists $y \in L$ such that $x = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$, because for any unitary $u \in N$,

$$\begin{pmatrix} u & 0 \\ 0 & \Phi(u) \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & \Phi(u^*) \end{pmatrix} = \begin{pmatrix} 0 & u\Phi(u^*) \\ 0 & 0 \end{pmatrix}.$$

Furthermore,

$$\|y - I\| \leq \sup_{u \in N^u} \|u\Phi(u^*) - I\| = \sup_{u \in N^u} \|\Phi(u^*) - u^*\| \leq \|\Phi - id_N\| < 1.$$

By Lemma 2.3, the unitary $u \in L$ in the polar decomposition $y = u|y|$ satisfies

$$\|u - I\| \leq \sqrt{2}\|\Phi - id_N\| < 20d(N, M).$$

Since $x = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in K'$, for any $n \in N$,

$$\begin{pmatrix} 0 & y\Phi(n) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} n & 0 \\ 0 & \Phi(n) \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & \Phi(n) \end{pmatrix} \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ny \\ 0 & 0 \end{pmatrix}.$$

By taking adjoints, we have

$$\Phi(n)y^* = y^*n \quad \text{for } n \in N.$$

Therefore,

$$y^*y\Phi(n) = y^*ny = \Phi(n)y^*y \quad \text{for } n \in N.$$

For any $n \in N$, since this gives $|y|\Phi(n) = \Phi(n)|y|$, $u\Phi(n) = nu$. Hence, $uMu^* = u\Phi(N)u^* = N$. \square

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